

On Monogamy of four qubit entanglement

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Our main result is a monogamy inequality satisfied by the entanglement of a focus qubit (one-tangle) in a four-qubit pure state and entanglement of subsystems. Analytical relations between three-tangles and two-tangles which quantify the entanglement of marginal states and unitary invariants of four-qubit pure state, are used to obtain the inequality. The contribution of three-tangle to one-tangle is found to be half of that suggested by a simple extension of entanglement monogamy relation for three qubits. On the other hand, an additional contribution due to a two-qubit invariant which is a function of three-way correlations, is found. We also show that four qubit monogamy inequality conjecture of ref. [PRL 113, 110501 (2014)] in which three-tangles are raised to the power $3/2$, does not estimate the residual correlations, correctly.

I. INTRODUCTION

Entanglement is a necessary ingredient of any quantum computation and a physical resource for quantum cryptography and quantum communication [1]. It has also found applications in other areas such as quantum field theory [2], statistical physics [3], and quantum biology [4]. Multipartite entanglement that comes into play in quantum systems with more than two subsystems, is a resource for multiuser quantum information tasks. Since the mathematical structure of multipartite states is much more complex than that of bipartite states, the characterization of multipartite entanglement is a far more challenging task [5].

Monogamy is a unique feature of quantum entanglement, which determines how entanglement is distributed amongst the subsystems. Three-qubit entanglement is known to satisfy a quantitative constraint, known as CKW monogamy inequality [6]. In recent articles [7–9], it has been shown that the most natural extension of CKW inequality to four-qubit entanglement is violated by some of the four-qubit states. To solve the problem, different ways to extend the monogamy inequality valid for all four-qubit states, have been conjectured [7, 9]. Three-qubit states show two distinct types of entanglement. As we go to four qubits, additional degrees of freedom make it possible for new entanglement types to emerge. It is signalled by the fact that corresponding to the three-qubit invariant that detects genuine three-way entanglement of a three-qubit pure state, a four-qubit pure state has five three-qubit invariants for each set of three qubits [10]. An m -qubit invariant is understood to be a function of state coefficients which remains invariant under the action of a local unitary transformation on the state of any one of the m qubits. A valid discussion of entanglement monogamy for four qubits must, therefore, include contributions from invariants that detect new entanglement types.

This letter is an attempt to identify, analytically, the contributions of two-tangles (pairwise entanglement), three-tangles (genuine three-way entanglement) and four tangles to entanglement of a focus qubit with the three remaining qubits (one-tangle) in a four qubit state. To do this, we express one-tangle in terms of two-qubit invariants. Monogamy inequality constraint on four qubit entanglement is obtained by comparing the one-tangle with upper bounds on two-tangles and three-tangles [11] defined on two and three qubit marginal states. Contribution of three-tangles to one-tangle is found to be half of what is expected from a direct generalization of CKW inequality to four qubits. The difference arises due to new entanglement modes that are available to four qubits. It is verified that the "residual entanglement", obtained after subtracting the contributions of two-tangles and three-tangles from one-tangle, is greater or equal to genuine four-tangle. Genuine four-tangle [10, 12] is a degree eight function of state coefficients of the pure state. Besides that, the "residual entanglement" contains a contribution from square of degree two four-tangle [13, 14].

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II. ONE-TANGLE OF A FOCUS QUBIT IN A FOUR-QUBIT STATE

Entanglement of qubit A_1 with qubit A_2 in pure state, $|\Psi_{12}\rangle = \sum_{i_1, i_2} a_{i_1 i_2} |i_1 i_2\rangle$ ($i_m = 0, 1$), is quantified by two-tangle defined as $\tau_{A_1|A_2}(|\Psi_{12}\rangle) = 2|D^{00}|$, where $D^{00} = a_{00}a_{11} - a_{10}a_{01}$ is a two-qubit invariant. On the other hand, for each choice of a pair of qubits one identifies nine two-qubit invariants on a four qubit pure state. Three-tangle [6] of a three-qubit pure state is defined in terms of modulus of a three-qubit invariant. On the most general four qubit state, however, we have five three-qubit invariants corresponding to a given choice of three qubits. Four qubit invariant that quantifies the sum of three-way and four-way correlations of a three-qubit partition in a pure state is known to be a degree eight invariant [10], which is a function of three qubit invariants. It is natural to expect that the monogamy inequality for four qubits takes into account entanglement modes available exclusively to four qubit system. To understand, how various two-tangles and three-tangles add up to generate total entanglement of a focus qubit in a pure four qubit state, we follow the steps listed below:

- (1) Write down one-tangle of focus qubit as a sum of two-qubit invariants.
- (2) Express two-tangles, three-tangles and four-tangle or the upper bounds on the tangles in terms of two-qubit invariants.
- (3) Compare and write down the exact relation between one-tangle and tangles defined on two and three qubit reduced states.

To facilitate the identification of two-qubit and three-qubit invariants, we use the formalism of determinants of two by two matrices of state coefficients referred to as negativity fonts. For more on definition and physical meaning of determinants of negativity fonts, please refer to section (VI) of ref. [10]. For the purpose of this article, we write down the determinants of negativity fonts of a four qubit state with qubit A_1 as the focus qubit. On a four-qubit pure state, written as

$$|\Psi_{1234}\rangle = \sum_{i_1, i_2, i_3, i_4} a_{i_1 i_2 i_3 i_4} |i_1 i_2 i_3 i_4\rangle, \quad (i_m = 0, 1), \quad (1)$$

where state coefficients $a_{i_1 i_2 i_3 i_4}$ are complex numbers and i_m refers to basis state of qubit A_m , ($m = 1, 2, 3$), we identify determinants of two-way negativity fonts to be $D_{(A_3)_{i_3}(A_4)_{i_4}}^{00} = a_{00i_3i_4}a_{11i_3i_4} - a_{10i_3i_4}a_{01i_3i_4}$, $D_{(A_2)_{i_2}(A_4)_{i_4}}^{00} = a_{0i_20i_4}a_{1i_21i_4} - a_{1i_20i_4}a_{0i_21i_4}$, and $D_{(A_2)_{i_2}(A_3)_{i_3}}^{00} = a_{0i_2i_30}a_{1i_2i_31} - a_{1i_2i_30}a_{0i_2i_31}$. Besides that we also have $D_{(A_4)_{i_4}}^{00i_3} = a_{00i_3i_4}a_{11,i_3\oplus 1,i_4} - a_{10i_3i_4}a_{01,i_3\oplus 1,i_4}$ (three-way), $D_{(A_3)_{i_3}}^{00i_4} = a_{00i_3i_4}a_{11i_3,i_4\oplus 1} - a_{10i_3i_4}a_{01i_3,i_4\oplus 1}$ (three-way), $D_{(A_2)_{i_2}}^{00i_4} = a_{0i_20i_4}a_{1i_21i_4\oplus 1} - a_{1i_20i_4}a_{0i_21i_4\oplus 1}$ (three-way), and $D^{00i_3i_4} = a_{00i_3i_4}a_{11,i_3\oplus 1,i_4\oplus 1} - a_{10i_3i_4}a_{01,i_3\oplus 1,i_4\oplus 1}$ (four-way), as the determinants of negativity fonts.

One-tangle given by $\tau_{1|234}(|\Psi_{1234}\rangle) = 4 \det(\rho_1)$, where $\rho_1 = \text{Tr}_{A_2 A_3 A_4}(|\Psi_{1234}\rangle\langle\Psi_{1234}|)$, quantifies the entanglement of qubit A_1 with A_2 , A_3 and A_4 . It is four times the square of negativity of partial transpose of four-qubit pure state with respect to qubit A_1 [17]. It is easily verified that

$$\begin{aligned} \tau_{1|234}(|\Psi_{1234}\rangle) = 4 & \left[\sum_{i_3, i_4=0}^1 \left| D_{(A_3)_{i_3}(A_4)_{i_4}}^{00} \right|^2 + \sum_{i_2, i_4=0}^1 \left| D_{(A_2)_{i_2}(A_4)_{i_4}}^{00} \right|^2 + \sum_{i_2, i_3=0}^1 \left| D_{(A_2)_{i_2}(A_3)_{i_3}}^{00} \right|^2 \right. \\ & \left. + \sum_{i_3, i_4=0}^1 \left| D_{(A_4)_{i_4}}^{00i_3} \right|^2 + \sum_{i_3, i_4=0}^1 \left| D_{(A_3)_{i_3}}^{00i_4} \right|^2 + \sum_{i_2, i_4=0}^1 \left| D_{(A_2)_{i_2}}^{00i_4} \right|^2 + \sum_{i_3, i_4=0}^1 \left| D^{00i_3i_4} \right|^2 \right]. \quad (2) \end{aligned}$$

One-tangle depends on 2-way, 3-way and 4-way correlations of focus qubit A_1 with the rest of the system.

III. DEFINITIONS OF TWO-TANGLES AND THREE-TANGLE

This section includes the definitions of two-tangles and three-tangles for pure and mixed three-qubit states. Consider a three-qubit pure state

$$|\Psi_{123}\rangle = \sum_{i_1, i_2, i_3} a_{i_1 i_2 i_3} |i_1 i_2 i_3\rangle, \quad i_m = 0, 1. \quad (3)$$

Using the notation from ref. [10], we define $D_{(A_3)_{i_3}}^{00} = a_{00i_3}a_{11i_3} - a_{10i_3}a_{01i_3}$, ($i_3 = 0, 1$) (determinant of a two-way negativity font) and $D^{00i_3} = a_{00i_3}a_{11i_3+1} - a_{10i_3}a_{01i_3+1}$, ($i_3 = 0, 1$) (determinant of a three-way negativity font).

Entanglement of qubit A_1 with the rest of the system is quantified by one-tangle $\tau_{1|23}(|\Psi_{123}\rangle) = 4 \det(\rho_1)$, where $\rho_1 = \text{Tr}_{A_2 A_3}(|\Psi_{123}\rangle\langle\Psi_{123}|)$. Three tangle of pure state $|\Psi_{123}\rangle$, defined in ref. [6], is equal to the modulus of a polynomial invariant of degree four that is $\tau_{1|2|3}(|\Psi_{123}\rangle) = 4 |I_{3,4}(|\Psi_{123}\rangle)|$, where

$$I_{3,4} = (D^{000} + D^{001})^2 - 4D_{(A_3)_0}^{00} D_{(A_3)_1}^{00}. \quad (4)$$

The entanglement measure $\tau_{1|2|3}(|\Psi_{123}\rangle)$ is extended to a mixed state of three qubits via convex roof extension that is

$$[\tau_{1|2|3}(\rho_{123})]^{\frac{1}{2}} = \min_{\{p_i, |\phi_{123}^i\rangle\}} \sum_i p_i [\tau_{1|2|3}(|\phi_{123}^i\rangle)]^{\frac{1}{2}}, \quad (5)$$

where minimization is taken over all complex decompositions $\{p_i, |\phi_{123}^i\rangle\}$ of ρ_{123} .

For qubit pair $A_1 A_2$ in the state $|\Psi_{123}\rangle$, we identify three two-qubit invariants that is

$$D_{(A_3)_0}^{00}, \frac{(D^{000} + D^{001})}{2}, D_{(A_3)_1}^{00} \quad (6)$$

while for the pair $A_1 A_3$ two-qubit invariants are

$$D_{(A_2)_0}^{00}, \frac{(D^{000} - D^{001})}{2}, D_{(A_2)_1}^{00}. \quad (7)$$

Two- tangle of state $\rho_{12} = \sum_i p_i |\phi_{12}^i\rangle\langle\phi_{12}^i|$ is constructed through convex roof extension as

$$\tau_{1|2}(\rho_{12}) = 2 \min_{\{p_i, \phi_i\}} \sum_i p_i |D^{00}(\phi_{12}^i)|. \quad (8)$$

Two-tangle $\tau_{1|2}(\rho_{12}) = C(\rho_{12})$, where $C(\rho_{12})$ is the concurrence [15, 16]. One can use formalism of determinants of negativity fonts to verify (Appendix A) that

$$\tau_{1|2|3}(|\Psi_{123}\rangle) = \tau_{1|23}(|\Psi_{123}\rangle) - [\tau_{1|2}(\rho_{12})]^2 - [\tau_{1|3}(\rho_{13})]^2. \quad (9)$$

Moduli of two-qubit invariants, which depends only on determinants of three-way negativity fonts, are used to define new two tangles that is

$$\tau_{1|2}^{(new)}(|\Psi_{123}\rangle) = 2 |D^{000}(|\Psi_{123}\rangle) + D^{001}(|\Psi_{123}\rangle)|, \quad (10)$$

and

$$\tau_{1|3}^{(new)}(|\Psi_{123}\rangle) = 2 |D^{000}(|\Psi_{123}\rangle) - D^{001}(|\Psi_{123}\rangle)|, \quad (11)$$

and extended to mixed state $\rho_{123} = \sum_i p_i |\phi_{123}^i\rangle\langle\phi_{123}^i|$ via

$$\tau_{1|p}^{(new)}(\rho_{123}) = 2 \min_{\{p_i, \phi_i\}} \sum_i p_i \tau_{1|p}^{(new)}(|\phi_{123}^i\rangle). \quad (12)$$

IV. TANGLES AND THREE-QUBIT INVARIANTS OF A FOUR QUBIT STATE

For a given pair of qubits in a four-qubit state, there are nine two-qubit invariants. Of the six three qubit invariants constructed from these, one is defined only on the pure state. Five remaining invariants are related to three-tangles, and two-tangles. In Table I, we identify sets of two-qubit invariants of a four-qubit state which transform under a unitary, $U = \frac{1}{\sqrt{1+|x|^2}} \begin{bmatrix} 1 & -x^* \\ x & 1 \end{bmatrix}$ on third qubit in the same way as the functions $A(x)$, $B(x)$ and $C(x)$ of Appendix B. Three qubit invariants listed in last three columns depend on two-qubit invariants of columns one to three in the same way as $I^{(1)}$, $I^{(2)}$ and τ depend on A , B and C , for example three qubit invariants in the first row of Table I read as

$$N_{A_4}^{(i)} = \left| D_{(A_3)_0(A_4)_i}^{00} \right|^2 + \frac{1}{2} \left| D_{(A_4)_i}^{000} + D_{(A_4)_i}^{001} \right|^2 + \left| D_{(A_3)_1(A_4)_i}^{00} \right|^2, \quad (13)$$

t

TABLE I: Two tangles and three tangles in terms of two-qubit invariants of a four qubit pure state ($i = 0, 1$). Here $I^{(1)} = |A|^2 + \frac{1}{2}|B|^2 + |C|^2$, $I^{(2)} = |B^2 - 4AC|$ and $\tau^2 = 4 \left(I^{(1)} - \frac{1}{2}I^{(2)} \right)$ (Appendix B)

		Two qubit invariants			Three qubit invariants		
Qubit pair	\downarrow	A	B	C	$I^{(1)}$	$I^{(2)}$	τ^2
$A_1 A_2$		$D_{(A_3)_0(A_4)_i}^{00}$	$D_{(A_4)_i}^{000} + D_{(A_4)_i}^{001}$	$D_{(A_3)_1(A_4)_i}^{00}$	$N_{A_4}^{(i)}$	$ I_{3,4}(\left \Phi_{123}^{(i)}\right\rangle) $	$\left[\tau_{1 2}^{(i)}(\left \Phi_{123}^{(i)}\right\rangle)\right]^2$
$A_1 A_2$		$D_{(A_3)_0}^{000} + D_{(A_3)_0}^{001}$	$\begin{pmatrix} D_{(A_3)_0}^{0000} + D_{(A_3)_0}^{0010} \\ + D_{(A_3)_0}^{0001} + D_{(A_3)_0}^{0011} \end{pmatrix}$	$D_{(A_3)_1}^{000} + D_{(A_3)_1}^{001}$	M_{A_3}	$ (I_3)_{A_4}^{(new)}(\left \Psi_{1234}\right\rangle) $	$\left[\tau_{1 2}^{(new)}(\rho_{124})\right]^2$
$A_1 A_3$		$D_{(A_2)_i(A_4)_0}^{00}$	$D_{(A_2)_i}^{000} + D_{(A_2)_i}^{001}$	$D_{(A_2)_i(A_4)_1}^{00}$	$N_{A_2}^{(i)}$	$ I_{3,4}(\left \Phi_{134}^{(i)}\right\rangle) $	$\left[\tau_{1 3}^{(i)}(\left \Phi_{134}^{(i)}\right\rangle)\right]^2$
$A_1 A_3$		$D_{(A_4)_0}^{000} - D_{(A_4)_0}^{001}$	$\begin{pmatrix} D_{(A_4)_0}^{0000} + D_{(A_4)_0}^{0001} \\ - D_{(A_4)_0}^{0010} - D_{(A_4)_0}^{0011} \end{pmatrix}$	$D_{(A_4)_1}^{000} - D_{(A_4)_1}^{001}$	M_{A_4}	$ (I_3)_{A_2}^{(new)}(\left \Psi_{1234}\right\rangle) $	$\left[\tau_{1 3}^{(new)}(\rho_{123})\right]^2$
$A_1 A_4$		$D_{(A_2)_0(A_3)_i}^{00}$	$D_{(A_3)_i}^{000} - D_{(A_3)_i}^{001}$	$D_{(A_2)_1(A_3)_i}^{00}$	$N_{A_3}^{(i)}$	$ I_{3,4}(\left \Phi_{124}^{(i)}\right\rangle) $	$\left[\tau_{1 4}^{(i)}(\left \Phi_{134}^{(i)}\right\rangle)\right]^2$
$A_1 A_4$		$D_{(A_2)_0}^{000} - D_{(A_2)_0}^{001}$	$\begin{pmatrix} D_{(A_2)_0}^{0000} + D_{(A_2)_0}^{0010} \\ - D_{(A_2)_0}^{0001} - D_{(A_2)_0}^{0011} \end{pmatrix}$	$D_{(A_2)_1}^{000} - D_{(A_2)_1}^{001}$	M_{A_2}	$ (I_3)_{A_3}^{(new)}(\left \Psi_{1234}\right\rangle) $	$\left[\tau_{1 4}^{(new)}(\rho_{134})\right]^2$

$$|I_{3,4}(\left|\Phi_{123}^{(i)}\right\rangle)| = \left| \left(D_{(A_4)_i}^{000} + D_{(A_4)_i}^{001} \right)^2 - 4D_{(A_3)_0(A_4)_i}^{00} D_{(A_3)_1(A_4)_i}^{00} \right|, \quad (14)$$

whereas

$$4 \sum_{i=0,1} N_{A_4}^{(i)} \geq \sum_{i=0,1} \left[\tau_{1|2}^{(i)}(\left|\Phi_{123}^{(i)}\right\rangle) \right]^2 + \frac{1}{2} \sum_{i=0,1} \left| 4I_{3,4}(\left|\Phi_{123}^{(i)}\right\rangle) \right|. \quad (15)$$

Here $\left|\Phi_{123}^{(i)}\right\rangle$ is the unnormalized state defined by $|\Psi_{1234}\rangle = \sum_{i_4=0,1} \left|\Phi_{123}^{(i_4)}\right\rangle |i_4\rangle$.

Upper bound on two-tangle calculated by using the method of ref. [10] shows that

$$\sum_{i=0,1} \left[\tau_{1|2}^{(i)}(\left|\Phi_{123}^{(i)}\right\rangle) \right]^2 \geq \left[\tau_{1|2}^{up}(\rho_{12}) \right]^2 \geq \left[\tau_{1|2}(\rho_{12}) \right]^2. \quad (16)$$

Similarly the upper bounds on $\tau_{1|2|3}(\rho_{123})$ for the nine families of four-qubit states, calculated in ref. ([11]) satisfy the condition

$$\sum_{i=0,1} \left| 4I_{3,4}(\left|\Phi_{123}^{(i)}\right\rangle) \right| \geq \tau_{1|2|3}^{up}(\rho_{123}) \geq \tau_{1|2|3}(\rho_{123}). \quad (17)$$

Combining the conditions of Eqs. (16) and (17), with inequality of Eq. (15), the sum of two tangle and three tangle satisfies the inequality

$$4 \sum_{i=0,1} N_{A_4}^{(i)} \geq \left[\tau_{1|2}(\rho_{12}) \right]^2 + \frac{1}{2} \tau_{1|2|3}(\rho_{123}). \quad (18)$$

On $\rho_{124} = \sum_i |\Phi_{124}^i\rangle \langle \Phi_{124}^i|$, new two-qubit invariant (Eq. (10)) is defined as

$$\tau_{1|2}^{(new)}(\rho_{124}) = \min_{\left\{ \left|\Phi_{124}^{(i)}\right\rangle \right\}} \left(\tau_{1|2}^{(new)}(\left|\Phi_{124}^{(0)}\right\rangle) + \tau_{1|2}^{(new)}(\left|\Phi_{124}^{(1)}\right\rangle) \right).$$

where $\tau_{1|2}^{(new)}(\left|\Phi_{124}^{(i)}\right\rangle) = 2 \left| D_{(A_3)_i}^{000}(\left|\Phi_{124}^{(i)}\right\rangle) + D_{(A_3)_i}^{001}(\left|\Phi_{124}^{(i)}\right\rangle) \right|$. New three-qubit tangle on a pure state is defined as $\tau_{1|2|3}^{(new)}(|\Psi_{1234}\rangle) = 4 \left| (I_3)_{A_4}^{(new)}(|\Psi_{1234}\rangle) \right|$. The invariants M_{A_3} , $\tau_{1|2|3}^{(new)}(|\Psi_{1234}\rangle)$ and $\left[\tau_{1|2}^{(new)}(\rho_{124}) \right]^2$ satisfy the inequality (analogous to Eq. (B7)),

$$4M_{A_3} \geq \left[\tau_{1|2}^{(new)}(\rho_{124}) \right]^2 + \frac{1}{2} \tau_{1|2|3}^{(new)}(|\Psi_{1234}\rangle). \quad (19)$$

Using a similar argument, three-qubit invariants listed in lines 4 and 5 of Table I satisfy the inequalities

$$4 \sum_i N_{A_2}^{(i)} \geq [\tau_{1|3}(\rho_{13})]^2 + \frac{1}{2} \tau_{1|3|4}(\rho_{134}), \quad (20)$$

and

$$4M_{A_4} \geq [\tau_{1|3}^{(new)}(\rho_{123})]^2 + \frac{1}{2} \tau_{1|3|4}^{(new)}(|\Psi_{1234}\rangle), \quad (21)$$

where three-tangle defined on pure four-qubit state reads as $\tau_{1|3|4}^{(new)}(|\Psi_{1234}\rangle) = 4 \left| (I_3)_{A_2}^{(new)}(|\Psi_{1234}\rangle) \right|$, and

$$\tau_{1|3}^{(new)}(\rho_{123}) = \min_{\{|\Phi_{123}^i\rangle\}} \sum_i \tau_{1|3}^{(new)}(|\Phi_{123}^i\rangle).$$

Using invariants of local unitaries on qubits A_1 and A_4 , and definitions given in lines 6 and 7 of Table I, we obtain the inequalities

$$4 \sum_i N_{A_3}^{(i)} \geq [\tau_{1|4}(\rho_{14})]^2 + \frac{1}{2} \tau_{1|2|4}(\rho_{124}) \quad (22)$$

and

$$4M_{A_2} \geq [\tau_{1|4}^{(new)}(\rho_{134})]^2 + \frac{1}{2} \tau_{1|2|4}^{(new)}(|\Psi_{1234}\rangle), \quad (23)$$

where new three-tangle reads as $\tau_{1|2|4}^{(new)}(|\Psi_{1234}\rangle) = 4 \left| (I_3)_{A_3}^{(new)}(|\Psi_{1234}\rangle) \right|$, and

$$\tau_{1|4}^{(new)}(\rho_{134}) = 2 \min_{\{|\Phi_{134}^i\rangle\}} \sum_i \left| D_{(A_2)_i}^{000}(|\Phi_{134}^i\rangle) - D_{(A_2)_i}^{001}(|\Phi_{134}^i\rangle) \right|.$$

V. MONOGAMY OF FOUR QUBIT ENTANGLEMENT

Genuine four tangle, defined in ref. [10, 12] is a degree eight function of state coefficients. Another four qubit invariant that equals invariant H of degree two of ref. [13, 14], is known to have the form,

$$I_{4,2}(|\Psi_{1234}\rangle) = (D^{0000} - D^{0010} - D^{0001} + D^{0011}). \quad (24)$$

Four tangle defined as $\tau_{1|2|3|4}^{(0)}(|\Psi_{1234}\rangle) = 2 |I_{4,2}(|\Psi_{1234}\rangle)|$, is non zero on a GHZ state and vanishes on W-like states of four qubits. However, since $\tau_{1|2|3|4}^{(0)}(|\Psi_{1234}\rangle)$ fails to vanish on product of entangled states of two qubits, it is not a measure of genuine four-way entanglement. By direct substitution it is verified that the tangle of Eq. (2), can be rewritten as a sum of three-qubit invariants listed in column five of Table I and square of four-qubit invariant $\tau_{1|2|3|4}^{(0)}(|\Psi_{1234}\rangle)$ that is

$$\tau_{1|234} = 4 \sum_{q=2}^4 \sum_{i=0}^1 N_{A_q}^{(i)} + 2 \sum_{q=2}^4 M_{A_q} + \frac{1}{4} \left[\tau_{1|2|3|4}^{(0)}(|\Psi_{1234}\rangle) \right]^2. \quad (25)$$

By using the inequalities given by Eqs. (18- 23) in Eq. (25), one-tangle is found to satisfy the constraint

$$\begin{aligned} \tau_{1|234}(|\Psi_{1234}\rangle) &\geq \sum_{p=2}^4 [\tau_{1|p}(\rho_{1p})]^2 + \frac{1}{2} \sum_{\substack{(p,q)=2 \\ q>p}}^4 \tau_{1|p|q}(\rho_{1pq}) \\ &\quad + \frac{1}{2} [\tau_{1|2}^{(new)}(\rho_{123})]^2 + \frac{1}{2} [\tau_{1|3}^{(new)}(\rho_{134})]^2 + \frac{1}{2} [\tau_{1|4}^{(new)}(\rho_{124})]^2. \end{aligned} \quad (26)$$

This is the monogamy relation that governs the distribution of quantum entanglement in subsystems of a four-qubit state, when qubit A_1 is the focus qubit. Similar inequalities can be found for other possible choices of focus qubit.

Parameter Δ , which quantifies the residual correlations not accounted for by entanglement of reduced states, is defined as the difference between the entanglement of focus qubit A_1 with the rest of the system and contributions from sum of two-way and three-way tangles of focus qubit, that is

$$\begin{aligned} \Delta = & \tau_{1|234}(|\Psi_{1234}\rangle) - \sum_{p=2}^4 [\tau_{1|p}(\rho_{1p})]^2 - \frac{1}{2} \sum_{\substack{(p,q)=2 \\ q>p}}^4 \tau_{1|p|q}(\rho_{1pq}) \\ & - \frac{1}{2} [\tau_{1|2}^{(new)}(\rho_{123})]^2 - \frac{1}{2} [\tau_{1|3}^{(new)}(\rho_{134})]^2 - \frac{1}{2} [\tau_{1|4}^{(new)}(\rho_{124})]^2. \end{aligned} \quad (27)$$

A lower bound on Δ is given by

$$\delta(|\Psi_{1234}\rangle) = \frac{1}{4} \sum_{\substack{(p,q)=2 \\ q>p}}^4 \tau_{1|p|q}^{(new)}(|\Psi_{1234}\rangle) + \frac{1}{4} [\tau_{1|2|3|4}^{(0)}(|\Psi_{1234}\rangle)]^2,$$

which partially accounts for residual correlations. It is expected that Δ contains contributions not only from genuine four tangle, but also from new three tangles (defined on a four qubit pure state), as well as, degree-two four tangle. This explains, why Δ is not independent of the choice of focus qubit.

VI. STATES VIOLATING GENERAL MONOGAMY INEQUALITY

As mentioned in ref. [9] a natural extension of CKW inequality of to four qubit states reads as

$$\tau_{1|234}(|\Psi_{1234}\rangle) \geq \sum_{p=2}^4 [\tau_{1|p}(\rho_{1p})]^2 + \sum_{\substack{(p,q)=2 \\ q>p}}^4 \tau_{1|p|q}(\rho_{1pq}). \quad (28)$$

The authors in ref. ([7]) reported that analysis of arbitrary pure states $|\Psi_4\rangle$ of four-qubit systems shows that a subset of four qubit states violates the inequality of Eq. (28). They argue that numerical calculations do not support the general monogamy inequality therefore they conjecture a new monogamy inequality. With A_1 as focus qubit the Monogamy inequality of Eq. (9) in ref. [7] for four qubits reads as

$$\begin{aligned} \tau_{1|234}(|\Psi_{1234}\rangle) \geq & [\tau_{1|2}(\rho_{12})]^2 + [\tau_{1|3}(\rho_{13})]^2 + [\tau_{1|4}(\rho_{14})]^2 \\ & + [\tau_{1|2|3}(\rho_{123})]^{\frac{3}{2}} + [\tau_{1|2|4}(\rho_{124})]^{\frac{3}{2}} + [\tau_{1|3|4}(\rho_{134})]^{\frac{3}{2}}. \end{aligned} \quad (29)$$

Here three tangles are raised to the power $\frac{3}{2}$, so that the "residual four tangle" may not become negative. We denote the residual correlations, calculated from inequality of Eq. (28) by Δ_1 and that from inequality proposed in ref. [7] (Eq. (28)) by Δ_2 . In a more recent article [9] it is clarified that the states leading to violations of the strong monogamy inequality belong to the degenerate subclasses (with $a = c$ or $b = c$) of $|G_{abc}^{(2)}\rangle$ [18] defined as

$$\begin{aligned} |G_{abc}^{(2)}\rangle = & \frac{a+b}{2}(|0000\rangle + |1111\rangle) + \frac{a-b}{2}(|0011\rangle + |1100\rangle) \\ & + c(|0101\rangle + |1010\rangle) + |0110\rangle. \end{aligned} \quad (30)$$

For the choice $b = c = ia$ with $a \geq 0$, we obtain the class of single parameter states $|G_{a,ia,ia}^{(2)}\rangle$.

Figure I displays the parameters Δ_1 (red line), Δ_2 (orange line), and Δ (green line), for the states $|G_{a,ia,ia}^{(2)}\rangle$ ($0 \leq a \leq 5$). As expected Δ_1 becomes negative for a certain range of values of a , while Δ_2 , and Δ remain positive for the set of states being considered. Since three tangle varies from zero to one, and $(\tau_{A_1|A_p|A_q})^{\frac{3}{2}} > \frac{1}{2}\tau_{A_1|A_p|A_q}$ for $\tau_{A_1|A_p|A_q} > 0.25$, it is obvious, however, that the conjecture of ref. ([9]) either overestimates or underestimates the four-way correlations in classes of four-qubit states with finite three tangles. Figure I also shows a plot (blue line) of $\delta(|G_{a,ia,ia}^{(2)}\rangle)$. Analytical expressions for relevant tangles and residual correlations for the states $|G_{a,ia,ia}^{(2)}\rangle$ are given in Appendix C.

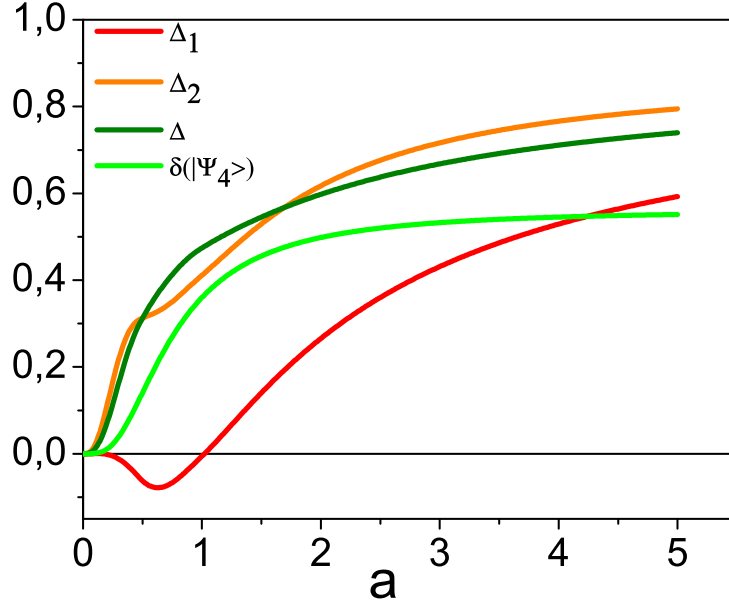


FIG. 1: Plot of Δ_1 (red line) from monogamy inequality of Eq. (28), Δ_2 (orange line) using monogamy conjecture of ref. [7], and Δ (green line) from our monogamy relation Eq. (26), for the states $|G_{a,ia,ia}^{(2)}\rangle$ ($0 \leq a \leq 5$). Figure also shows a plot (blue line) of $\Delta(\Psi_4)$

VII. CONCLUSIONS

Monogamy inequality of Eq. (26), obtained analytically by expressing the tangle in terms of two-qubit and three-qubit invariants, is the correct inequality satisfied by all classes of four-qubit states. The residual entanglement, obtained by subtracting the contributions of two-tangles and three-tangles from one-tangle of focus qubit does not represent genuine four-tangle, which is a function of a degree eight polynomial invariant. On pure states $|G_{abc}^{(2)}\rangle$ with $b = c = ia$, value of genuine four-tangle is found to be zero, while the residual entanglement (Δ) is non zero. Since in the states $|G_{a,ia,ia}^{(2)}\rangle$ all three tangles are finite for $a \neq 0$, four qubits are entangled to each other through three-way correlations. Four-qubit states may, likewise, have entanglement due to combination of three-way and two-way, or only pairwise correlations.

The monogamy inequality conjecture for four qubits [7] in which quantifiers of three-way entanglement (three tangles) should be raised to the power $\frac{3}{2}$ does not estimate the residual correlations, correctly. A simple example is state $|\Psi\rangle = |0000\rangle + |0101\rangle + |1000\rangle + |1110\rangle$, for which $\Delta = 0$, but Eq. (29) predicts $\Delta_2 = \frac{3}{8}$. In general, we disagree with the approach of ref. [7], where an n -tangle is raised to an arbitrary power to account for correlations represented by a one-tangle. A simple calculation, on the same lines as for four qubits, shows that as more qubits are added the contribution to one-tangle from tangles defined on three-qubit pure states is always multiplied by a factor of $\frac{1}{2}$. As new entanglement modes arise with an increase in the number of qubits, the contribution of degree four n -tangle ($\tau_{1|2|\dots|n}$) to degree four one-tangle will be multiplied by a factor of $(\frac{1}{2})^{n-2}$. Our approach paves the way to understanding scaling of entanglement distribution as qubits are added to obtain larger multiqubit quantum systems..

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Appendix A: Monogamy of three qubit entanglement

In this section formalism of determinants of negativity fonts is used to arrive at CKW inequality. Consider a three-qubit pure state

$$|\Psi_{123}\rangle = \sum_{i_1, i_2, i_3} a_{i_1 i_2 i_3} |i_1 i_2 i_3\rangle, \quad i_m = 0, 1, \quad (\text{A1})$$

where state coefficients $a_{i_1 i_2 i_3}$ are complex numbers and i_m refers to basis state of qubit A_m , ($m = 1, 2, 3$). Using the notation from ref. [10], we define $D_{(A_3)_{i_3}}^{00} = a_{00i_3} a_{11i_3} - a_{10i_3} a_{01i_3}$, ($i_3 = 0, 1$) (determinant of a two-way negativity font) and $D^{00i_3} = a_{00i_3} a_{11i_3+1} - a_{10i_3} a_{01i_3+1}$, ($i_3 = 0, 1$) (determinant of a three-way negativity font). Entanglement of qubit A_1 with the rest of the system is quantified by tangle $\tau_{1|23}(|\Psi_3\rangle) = 4 \det(\rho^{A_1})$. Three tangle of pure state $|\Psi_{123}\rangle$, defined in ref. [6], is equal to the modulus of a polynomial invariant of degree four that is

$$\tau_{1|2|3}(|\Psi_{123}\rangle) = 4 |I_{3,4}(|\Psi_{123}\rangle)|.$$

where

$$I_{3,4} = (D^{000} + D^{001})^2 - 4D_{(A_3)_0}^{00} D_{(A_3)_1}^{00}. \quad (\text{A2})$$

The entanglement measure $\tau_{1|2|3}(|\Psi_{123}\rangle)$ is extended to a mixed state of three qubits via convex roof extension that is

$$[\tau_{1|2|3}(\rho_{123})]^{\frac{1}{2}} = \min_{\{p_i, |\Phi_i\rangle\}} \sum_i p_i [\tau_{1|2|3}(|\Phi_{123}^i\rangle)]^{\frac{1}{2}}, \quad (\text{A3})$$

where minimization is taken over all possible decompositions $\{p_i, |\Phi_{123}^i\rangle\}$ of ρ_{123} .

For qubit pair $A_1 A_2$ of state $|\Psi_{123}\rangle$, we identify three two-qubit invariants that is

$$D_{(A_3)_0}^{00}, \frac{(D^{000} + D^{001})}{2}, D_{(A_3)_1}^{00}, \quad (\text{A4})$$

while for the pair $A_1 A_3$ two-qubit invariants are

$$D_{(A_2)_0}^{00}, \frac{(D^{000} - D^{001})}{2}, D_{(A_2)_1}^{00}. \quad (\text{A5})$$

Two-tangle $\tau_{1|2}(\rho_{12}) = C(\rho_{12})$, where $C(\rho_{12})$ is the concurrence [15, 16].

Two-qubit invariants of Eqs. (A4) and (A5) transform under a unitary $U = \frac{1}{\sqrt{1+|x|^2}} \begin{bmatrix} 1 & -x^* \\ x & 1 \end{bmatrix}$ on the third qubit in a way analogous to that of complex functions A , B and C of Appendix B. Then the invariants corresponding to $I^{(1)}$, $I^{(2)}$ are three qubit invariants for the given choice of qubit pair. In Table I, we enlist the correspondence of two-qubit invariants for qubit pairs $A_1 A_2$ and $A_1 A_3$, with complex numbers A , B and C of Appendix B and set the notation for invariants corresponding to $I^{(1)}$, $I^{(2)}$ and I as well as two tangles and three tangle. Two-tangles and three tangle satisfy the inequalities (from Eq. B7),

$$4N_{A_3} = [\tau_{1|2}(\rho_{12})]^2 + \frac{1}{2} \tau_{1|2|3}(|\Psi_{123}\rangle), \quad (\text{A6})$$

TABLE II: Two-tangles and three tangle in terms of two-qubit invariants of a three qubit pure state. Here $I^{(1)} = |A|^2 + \frac{1}{2} |B|^2 + |C|^2$, $I^{(2)} = B^2 - 4AC$ and $I = 4 \left(I^{(1)} - \frac{1}{2} |I^{(2)}| \right)$ (Appendix B)

Qubit Pair	A	B	C	$I^{(1)}$	$I^{(2)}$	Three tangle= $4I^{(2)}$	Two tangle= I
$A_1 A_2$	$D_{(A_3)_0}^{00}$	$D^{000} + D^{001}$	$D_{(A_3)_1}^{00}$	N_{A_3}	$ I_{3,4}(\Psi_{123}\rangle) $	$\tau_{1 2 3}(\Psi_{123}\rangle)$	$\tau_{1 2}(\rho_{12})$
$A_1 A_3$	$D_{(A_2)_0}^{00}$	$D^{000} - D^{001}$	$D_{(A_2)_1}^{00}$	N_{A_2}	$ I_{3,4}(\Psi_{123}\rangle) $	$\tau_{1 2 3}(\Psi_{123}\rangle)$	$\tau_{1 3}(\rho_{13})$

and

$$4N_{A_2} = [\tau_{1|3}(\rho_{13})]^2 + \frac{1}{2}\tau_{1|2|3}(|\Psi_{123}\rangle). \quad (\text{A7})$$

One can also verify that $\tau_{1|23}(|\Psi_{123}\rangle) = 4(N_{A_2} + N_{A_3})$, therefore

$$\tau_{1|23}(|\Psi_{123}\rangle) = [\tau_{1|2}(\rho_{12})]^2 + [\tau_{1|3}(\rho_{13})]^2 + \tau_{1|2|3}(|\Psi_{123}\rangle)$$

or

$$\tau_{1|23}(|\Psi_{123}\rangle) \geq [\tau_{1|2}(\rho_{12})]^2 + [\tau_{1|3}(\rho_{13})]^2, \quad (\text{A8})$$

which is the well known CKW inequality.

Appendix B: Upper bound on a function of complex numbers

In this section, we obtain mathematical relations used in section IV to define necessary invariants on four-qubit and three-qubit states. Let $A(x)$, $B(x)$ and $C(x)$ satisfy the set of equations

$$A(x) = \frac{1}{1+|x|^2} \left(A - x^* B + (x^*)^2 C \right), \quad (\text{B1})$$

$$B(x) = \frac{1}{1+|x|^2} \left(B \left(1 - |x|^2 \right) - 2x^* C + 2xA \right), \quad (\text{B2})$$

$$C(x) = \frac{1}{1+|x|^2} \left(C + xB + x^2 A \right), \quad (\text{B3})$$

where A , B , and C are complex numbers. One can verify that

$$I^{(1)} = |A|^2 + \frac{1}{2}|B|^2 + |C|^2 = |A(x)|^2 + \frac{1}{2}|B(x)|^2 + |C(x)|^2. \quad (\text{B4})$$

Consider a function of complex variable x defined as

$$\tau = 2 \min_x (|A(x)| + |C(x)|) = \min_x I(x), \quad (\text{B5})$$

Value of x for which $A(x) = 0$ is

$$x_0^* = \frac{B}{2C} \pm \frac{1}{2C} \sqrt{B^2 - 4AC}.$$

The discriminant $B^2 - 4AC$ is, obviously, an invariant, so we define $I^{(2)} = |B^2 - 4AC|$. Substituting the value of x_0 in Eq. (B3), we obtain

$$I(x_0) = 2|C(x_0)| = 2\sqrt{I^{(1)} - \frac{1}{2}I^{(2)}}; I^{(1)} \geq \frac{1}{2}I^{(2)} \quad (\text{B6})$$

From the definition of τ , $I(x_0)$ is an upper bound on I that is

$$\tau^2 \leq 4I^{(1)} - 2I^{(2)}. \quad (\text{B7})$$

Appendix C: Residual Interaction for states $|G_{a,ia,ia}^{(2)}\rangle$

In a recent article [9] it has been clarified that the states leading to violations of the strong monogamy inequality belong to the degenerate subclasses (with $a = c$ or $b = c$) of $|G_{abc}^{(2)}\rangle$ [18], where

$$|G_{abc}^{(2)}\rangle = \frac{a+b}{2}(|0000\rangle + |1111\rangle) + \frac{a-b}{2}(|0011\rangle + |1100\rangle) + c(|0101\rangle + |1010\rangle) + |0110\rangle. \quad (C1)$$

In this section we calculate the two-qubit and three-qubit invariants on class of single parameter states $|G_{a,ia,ia}^{(2)}\rangle$, ($b = c = ia$ with $a \geq 0$). Firstly, determinants of four-way negativity fonts for the state are found to have the values, $D^{0000} = \frac{ia^2}{2}$, $D^{0011} = \frac{-ia^2}{2}$, $D^{0010} = a^2$ and $D^{0001} = 0$. Relevant sets of two-qubit invariants which transform as $A(x)$, $B(x)$ and $C(x)$ of Appendix B, are listed in Table II.

Analytical expressions for relevant tangles and residual correlations for the states $|G_{a,ia,ia}^{(2)}\rangle$ are given below. It is found that with qubit A_1 as the focus qubit, tangle takes the value

$$\tau_{1|234} = \frac{8a^2 + 16a^4}{(4a^2 + 1)^2}, \quad (C2)$$

while $\tau_{1|2|3|4}^{(0)} = \frac{2a^2}{(4a^2+1)}$. Two qubit states obtained from $|G_{a,ia,ia}^{(2)}\rangle$ after tracing out a pair of qubits are X states. Analytical expressions for, $[\tau_{1|p}]^2 = C^2(\rho_{1p})$ look like

$$[\tau_{1|2}]^2 = \frac{4a^2 + 8a^4 - 8a^3\sqrt{1+a^2}}{(4a^2+1)^2} \text{ for } 0 \leq a \leq 5 \quad (C3)$$

$$[\tau_{1|3}]^2 = [\tau_{1|4}]^2 = \frac{2a^2 + 10a^4 - 6a^3\sqrt{2+a^2}}{(4a^2+1)^2}; 0 \leq a \leq \frac{1}{2}$$

TABLE III: Sets of two-qubit invariants which transform as $A(x)$, $B(x)$ and $C(x)$ of Appendix B, for class of single parameter states $|G_{a,ia,ia}^{(2)}\rangle$.

Two qubit invariants				
Qubit pair ↓	Unitary	A	B	C
$A_1 A_2$	U^{A_3}	$D_{(A_3)_0(A_4)_0}^{00} = \frac{a^2}{2(4a^2+1)}$	$D_{(A_4)_0}^{000} + D_{(A_4)_0}^{001} = 0$	$D_{(A_3)_1(A_4)_0}^{00} = \frac{ia}{4a^2+1}$
$A_1 A_2$	U^{A_3}	$D_{(A_3)_0}^{000} + D_{(A_3)_0}^{001} = 0$	$(D_{(A_3)_0}^{0000} + D_{(A_3)_0}^{0010} + D_{(A_3)_0}^{0001} + D_{(A_3)_0}^{0011}) = \frac{a^2}{4a^2+1}$	$D_{(A_3)_1}^{000} + D_{(A_3)_1}^{001} = 0$
$A_1 A_2$	U^{A_3}	$D_{(A_3)_0(A_4)_1}^{00} = 0$	$D_{(A_4)_1}^{000} + D_{(A_4)_1}^{001} = 0$	$D_{(A_3)_1(A_4)_1}^{00} = \frac{a^2}{2(4a^2+1)}$
$A_1 A_3$	U^{A_4}	$D_{(A_2)_0(A_4)_0}^{00} = \frac{a^2(i-1)}{2(4a^2+1)}$	$D_{(A_2)_0}^{000} + D_{(A_2)_0}^{001} = 0$	$D_{(A_2)_0(A_4)_1}^{00} = 0$
$A_1 A_3$	U^{A_4}	$D_{(A_4)_0}^{000} - D_{(A_4)_0}^{001} = 0$	$(D_{(A_4)_0}^{0000} + D_{(A_4)_0}^{0001} - D_{(A_4)_0}^{0010} - D_{(A_4)_0}^{0011}) = \frac{a^2(i-1)}{2(4a^2+1)}$	$D_{(A_4)_1}^{000} - D_{(A_4)_1}^{001} = 0$
$A_1 A_3$	U^{A_4}	$D_{(A_2)_1(A_4)_0}^{00} = \frac{a^2(i-1)}{2(4a^2+1)}$	$D_{(A_2)_1}^{000} + D_{(A_2)_1}^{001} = 0$	$D_{(A_2)_1(A_4)_1}^{00} = \frac{a^2(i-1)}{2(4a^2+1)}$
$A_1 A_4$	U^{A_2}	$D_{(A_2)_0(A_3)_0}^{00} = 0$	$D_{(A_3)_0}^{000} - D_{(A_3)_0}^{001} = 0$	$D_{(A_2)_1(A_3)_0}^{00} = -\frac{a^2(i+1)}{2(4a^2+1)}$
$A_1 A_4$	U^{A_2}	$D_{(A_2)_0}^{000} - D_{(A_2)_0}^{001} = 0$	$(D_{(A_2)_0}^{0000} + D_{(A_2)_0}^{0010} - D_{(A_2)_0}^{0001} - D_{(A_2)_0}^{0011}) = \frac{a^2(i+1)}{2(4a^2+1)}$	$D_{(A_2)_1}^{000} - D_{(A_2)_1}^{001} = 0$
$A_1 A_4$	U^{A_2}	$D_{(A_2)_0(A_3)_1}^{00} = -\frac{a^2(i+1)}{2(4a^2+1)}$	$D_{(A_3)_1}^{000} - D_{(A_3)_1}^{001} = 0$	$D_{(A_2)_1(A_3)_1}^{00} = \frac{a^2(i+1)}{2(4a^2+1)}$

TABLE IV: Values of three tangles and two-tangles corresponding to nine sets of two-qubit invariants listed in Table II, for class of single parameter states $|G_{a,ia,ia}^{(2)}\rangle$. Here $I^{(1)} = |A|^2 + \frac{1}{2}|B|^2 + |C|^2$, $I^{(2)} = B^2 - 4AC$ and $\tau = 4\left(I^{(1)} - \frac{1}{2}|I^{(2)}|\right)$ (Appendix B)

Three qubit invariants and tangles for states $ G_{a,ia,ia}^{(2)}\rangle$.			
Qubit pair \downarrow	$I^{(1)} = A ^2 + \frac{1}{2} B ^2 + C ^2$	$I^{(2)} = B^2 - 4AC $	$\tau^2 = 4I^{(1)} - \frac{1}{2} 4I^{(2)} $
$A_1 A_2$	$4N_{A_4}^{(0)} = \frac{4a^2+a^4}{(4a^2+1)^2}$	$ I_{3,4}(\Phi_{123}^i\rangle) = \frac{2a^3}{(4a^2+1)^2}$ $\tau_{1 2 3}^{(0)}(\rho_{123}^0) = \frac{8a^3}{(4a^2+1)^2}$	$[\tau_{1 2}^{(0)}(\rho_{123}^0)]^2 = \frac{4a^2+a^4-4a^3}{(4a^2+1)^2}$ $= 4N_{A_4}^{(0)} - \frac{1}{2}\tau_{1 2 3}^{(0)}(\rho_{123}^0)$
$A_1 A_2$	$M_{A_3} = \frac{a^4}{2(4a^2+1)^2}$	$ (I_3)_{A_4}^{(new)}(\Psi_{1234}\rangle) = \frac{a^4}{(4a^2+1)^2}$ $\tau_{1 2 4}^{(new)}(\Psi_{1234}\rangle) = \frac{4a^4}{(4a^2+1)^2}$	$[\tau_{1 2}^{(new)}(\rho_{124})]^2 = 0$ $= 2M_{A_3} - \frac{1}{4}\tau_{1 2 4}^{(new)}(\Psi_{1234}\rangle)$
$A_1 A_2$	$N_{A_4}^{(1)} = \frac{a^4}{4(4a^2+1)^2}$	$ I_{3,4}(\Phi_{123}^i\rangle) = 0 = \tau_{1 2 3}^{(1)}(\rho_{123}^1)$	$[\tau_{1 2}^{(1)}(\rho_{123}^1)]^2 = \frac{a^4}{(4a^2+1)^2}$ $= 4N_{A_4}^{(1)}$
$A_1 A_3$	$N_{A_2}^{(0)} = \frac{a^4}{2(4a^2+1)^2}$	$ I_{3,4}(\Phi_{134}^i\rangle) = 0 = \tau_{1 3 4}^{(0)}(\rho_{134}^0)$	$[\tau_{1 3}^{(0)}(\rho_{134}^0)]^2 = \frac{2a^4}{(4a^2+1)^2}$ $= 4N_{A_2}^{(0)}$
$A_1 A_3$	$M_{A_4} = \frac{a^4}{(4a^2+1)^2}$	$ (I_3)_{A_2}^{(new)}(\Psi_{1234}\rangle) = \frac{2a^4}{(4a^2+1)^2}$ $\tau_{1 3 4}^{(new)}(\rho_{123}) = \frac{8a^4}{(4a^2+1)^2}$	$[\tau_{1 3}^{(new)}(\rho_{123})]^2 = 0$ $= 2M_{A_4} - \frac{1}{4}\tau_{1 3 4}^{(new)}(\Psi_{1234}\rangle)$
$A_1 A_3$	$N_{A_2}^{(1)} = \frac{a^2+a^4}{2(4a^2+1)^2}$	$ I_{3,4}(\Phi_{134}^1\rangle) = \frac{2a^3}{(4a^2+1)^2}$ $\tau_{1 3 4}^{(1)} = \frac{8a^3}{(4a^2+1)^2}$	$[\tau_{1 3}^{(1)}(\rho_{134}^1)]^2 = \frac{2a^2+2a^4-4a^3}{(4a^2+1)^2}$ $= 4N_{A_2}^{(1)} - \frac{1}{2}\tau_{1 3 4}^{(1)}$
$A_1 A_4$	$N_{A_3}^{(0)} = \frac{a^4}{2(4a^2+1)^2}$	$ I_{3,4}(\Phi_{124}^0\rangle) = 0 = \tau_{1 2 4}^{(0)}(\rho_{124}^0)$	$[\tau_{1 4}^{(0)}(\rho_{124}^0)]^2 = \frac{2a^4}{(4a^2+1)^2}$ $= 4N_{A_3}^{(0)}$
$A_1 A_4$	$M_{A_2} = \frac{a^4}{(4a^2+1)^2}$	$ (I_3)_{A_3}^{(new)}(\Psi_{1234}\rangle) = \frac{2a^4}{(4a^2+1)^2}$ $\tau_{1 2 4}^{(new)}(\Psi_{1234}\rangle) = \frac{8a^4}{(4a^2+1)^2}$	$[\tau_{1 4}^{(new)}(\rho_{134})]^2 = 0$ $= 2M_{A_2} - \frac{1}{4}\tau_{1 2 4}^{(new)}(\Psi_{1234}\rangle)$
$A_1 A_4$	$N_{A_3}^{(1)} = \frac{a^2+a^4}{2(4a^2+1)^2}$	$ I_{3,4}(\Phi_{124}^1\rangle) = \frac{2a^3}{(4a^2+1)^2}$ $\tau_{1 2 4}^{(1)}(\rho_{124}^1) = \frac{8a^3}{(4a^2+1)^2}$	$[\tau_{1 4}^{(1)}(\rho_{124}^1)]^2 = \frac{2a^2+2a^4-4a^3}{(4a^2+1)^2}$ $= 4N_{A_3}^{(1)} - \frac{1}{2}\tau_{1 2 4}^{(1)}(\rho_{124}^1)$

$$[\tau_{1|3}]^2 = [\tau_{1|4}]^2 = 0; \quad \frac{1}{2} \leq a \leq \frac{1}{\sqrt{2}} \quad (C4)$$

$$[\tau_{1|3}]^2 = [\tau_{1|4}]^2 = \frac{2a^2 + 4a^4 - 4a^3\sqrt{2}}{(4a^2 + 1)^2}; \quad \frac{1}{\sqrt{2}} \leq a \leq 5$$

One can verify that

$$[\tau_{1|2}^{(0)}(\rho_{123}^0)]^2 + [\tau_{1|2}^{(1)}(\rho_{123}^1)]^2 \geq [\tau_{1|2}(\rho_{12})]^2, \quad (C5)$$

and

$$[\tau_{1|3}^{(0)}(\rho_{134}^0)]^2 + [\tau_{1|3}^{(1)}(\rho_{134}^1)]^2 \geq [\tau_{1|3}(\rho_{13})]^2.$$

Three qubit invariants listed in Table III depend on two-qubit invariants of columns three to five of Table II in same way as $I^{(1)}$, $I^{(2)}$ and τ depend on A , B and C of Appendix B. Values of three-tangles and two-tangles for class of single parameter states $|G_{a,ia,ia}^{(2)}\rangle$ are also contained Table III. Three tangles for the states satisfy, $\tau_{1|2|3} = \tau_{1|2|4} = \tau_{1|3|4} = \frac{8a^3}{(4a^2+1)^2}$.

All new two-tangles take value zero, while the sum of new three tangles defined on pure four-qubit state is found to be

$$\frac{1}{4} \left(\tau_{1|2|3}^{(new)}(\Psi_4) + \tau_{1|2|4}^{(new)}(\Psi_4) + \tau_{1|3|4}^{(new)}(\Psi_4) \right) = \frac{5a^4}{(4a^2 + 1)^2}. \quad (C6)$$

The residual entanglement

$$\begin{aligned} \Delta &= \tau_{1|234} - [\tau_{1|2}(\rho_{12})]^2 - [\tau_{1|3}(\rho_{13})]^2 - [\tau_{1|4}(\rho_{14})]^2 \\ &\quad - \frac{1}{2}\tau_{1|2|3}(\rho_{123}) - \frac{1}{2}\tau_{1|2|4}(\rho_{124}) - \frac{1}{2}\tau_{1|3|4}(\rho_{134}), \end{aligned} \quad (C7)$$

for the state $|G_{a,ia,ia}^{(2)}\rangle$ reads as

$$\Delta = \frac{4a^3(2\sqrt{a^2+1} + 3\sqrt{a^2+2} - 3(1+a))}{(4a^2+1)^2}; \text{ for } 0 \leq a \leq \frac{1}{2}, \quad (C8)$$

$$\Delta = \frac{4a^3(2a - 3 + 2\sqrt{a^2+1} + \frac{1}{a})}{(4a^2+1)^2}; \text{ for } \frac{1}{2} \leq a \leq \frac{1}{\sqrt{2}}, \quad (C9)$$

$$\Delta = \frac{4a^3(2\sqrt{2} + 2\sqrt{a^2+1} - 3)}{(4a^2+1)^2}; \text{ for } \frac{1}{\sqrt{2}} \leq a \leq 5, \quad (C10)$$

while $\delta(|G_{a,ia,ia}^{(2)}\rangle) = \frac{6a^4}{(4a^2+1)^2}$ is the Lower Bound on Δ . The residual entanglement may be rewritten as

$$\Delta = \delta_{12} + \delta_{13} + \delta_{14} + \delta(|G_{a,ia,ia}^{(2)}\rangle)$$

where

$$\delta_{12} = [\tau_{1|2}^{(0)}(\rho_{123}^{(0)})]^2 + [\tau_{1|2}^{(1)}(\rho_{123}^{(1)})]^2 - [\tau_{A_1|A_2}(\rho_{12})]^2,$$

and

$$\delta_{13} = \delta_{14} = [\tau_{1|3}^{(0)}(\rho_{134}^{(0)})]^2 + [\tau_{1|3}^{(1)}(\rho_{134}^{(1)})]^2 - [\tau_{A_1|A_3}(\rho_{13})]^2.$$

Appendix D: Upper bound on a degree-two unitary invariant

Let $E(|\phi\rangle)$ be a unitarily invariant degree-two function of state coefficients of a pure state $|\phi\rangle$. A mixed state can be expressed as a convex sum of pure states as, $\rho = \sum_i p_i |\phi_i\rangle \langle \phi_i|$, however, the decomposition to pure states is not unique. An extension of $E(|\phi\rangle)$ to mixed states is made by a convex roof construction as

$$E(\rho) = \min_{\{p_i, \phi_i\}} \sum_i p_i E(|\phi_i\rangle), \quad p_i > 0, \quad (D1)$$

where $\{p_i, \phi_i\}$ is a decomposition to pure states. Since p_i and $E(|\phi_i\rangle)$ are degree-two functions of state coefficients, $p_i E(|\phi_i\rangle) = E(|\Phi_i\rangle)$, where $|\Phi_i\rangle = \sqrt{p_i} |\phi_i\rangle$, is a sub-normalized state. In this case we may rewrite $\rho = \sum_i |\Phi_i\rangle \langle \Phi_i|$ and convex roof extension reads as

$$E(\rho) = \min_{\{\Phi_i\}} \sum_i E(|\Phi_i\rangle).$$

Consider an N -qubit state $|\Psi_N\rangle = \sum_{i_{A_q}} |\Phi_{A_q}^0\rangle |i_{A_q}\rangle$, such that the reduced state obtained by tracing out qubit A_q is

$\rho_{N-1} = \sum_i |\Phi_{A_q}^i\rangle \langle \Phi_{A_q}^i|$, then

$$E(\rho) = \min_{\{|\Phi_{A_q}^i\rangle\}} \sum_{i=0}^1 E(|\Phi_{A_q}^i\rangle). \quad (D2)$$

A unitary $U = \frac{1}{\sqrt{1+|x|^2}} \begin{bmatrix} 1 & -x^* \\ x & 1 \end{bmatrix}$, on q^{th} qubit gives a new decomposition $\rho(x) = \sum_i \left| \Phi_{A_q}^i(x) \right\rangle \left\langle \Phi_{A_q}^i(x) \right|$. We may rewrite $E(\rho) = \min_{\{x\}} \sum_{i=1}^2 E\left(\left| \Phi_{A_q}^i(x) \right\rangle\right)$. To obtain an upper bound on $E(\rho)$, we find the value of complex parameter x_1 such that $E\left(\left| \Phi_{A_q}^0(x_1) \right\rangle\right) = 0$, and x_2 such that $E\left(\left| \Phi_{A_q}^1(x_2) \right\rangle\right) = 0$. By definition $E(\rho)$ satisfies the constraint

$$E(\rho) \leq \min \left[E\left(\left| \Phi_{A_q}^0(x_2) \right\rangle\right), E\left(\left| \Phi_{A_q}^1(x_1) \right\rangle\right) \right], \quad (D3)$$

giving an upper bound on $E(\rho)$. In this case,

$$E^2(\rho) \leq \min \left[E^2\left(\left| \Phi_{A_q}^0(x_2) \right\rangle\right), E^2\left(\left| \Phi_{A_q}^1(x_1) \right\rangle\right) \right], \quad (D4)$$

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